

INTERPOLATION HILBERT SPACES FOR A COUPLE OF SOBOLEV SPACES

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ABSTRACT. We describe all Hilbert spaces that are interpolation spaces with respect to a given couple of Sobolev inner product spaces over \mathbb{R}^n or a bounded domain with a smooth boundary. We prove that these interpolation spaces form a subclass of isotropic Hörmander spaces. They are parametrized with a radial function parameter which is *RO*-varying at $+\infty$, considered as a function of $(1 + |\xi|^2)^{1/2}$ with $\xi \in \mathbb{R}^n$.

1. INTRODUCTION

A fundamental importance of Sobolev spaces for analysis and the theory of partial differential equations is well known. This importance comes in part from interpolation properties of the Sobolev scale [1, 2]. They enable to extend a number of important properties that Sobolev spaces of integer order possess over spaces of fractional order, e.g., the invariance with respect to an admissible change of variables. In its turn, this permits to define correctly a Sobolev space of any order over a smooth manifold.

At the same time, the scale of Sobolev spaces is not sufficiently fine for a number of mathematical problems. This explains a natural need to replace them with more general isotropic Hörmander spaces $H^{\varphi(\cdot)}$ [3]. They are initially defined over \mathbb{R}^n using the Fourier transform and a corresponding radial weight function $\varphi(\cdot)$ of the scalar argument $\langle \xi \rangle := (1 + |\xi|^2)^{1/2} \geq 1$. The use of a power function $\varphi(t) = t^s$ leads to the Sobolev space $H^{(s)}$. The application of an interpolation with a function parameter allows to transfer completely the classical theory of elliptic boundary-value problems from the Sobolev scale to a more extensive scale of Hörmander inner product spaces. The latter scale is parametrized with the function $\varphi(\cdot)$ that varies regularly in J. Karamata's sense at $+\infty$ with an arbitrary index $s \in \mathbb{R}$ [4–7].

In contrast to the case of spaces of integrable functions, interpolation of spaces of differentiable functions using a general function parameter has not been sufficiently studied even if Hilbert spaces are used [8, 9, 10]. The purpose of this paper is to describe constructively all Hilbert function spaces that are interpolation spaces with respect to a given couple of Sobolev inner product spaces

$$(1) \quad \{H^{(s_0)}, H^{(s_1)}\}, \quad -\infty < s_0 < s_1 < +\infty,$$

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over \mathbb{R}^n or a bounded Euclidean domain. We show that these spaces form a subclass of isotropic Hörmander spaces. This subclass is characterized by the fact that the function $\varphi(\cdot)$ is *RO*-varying at $+\infty$ in the sense of V. G. Avakumović [11, 12] however it can cease to be regularly varying. The chosen class of function spaces is sufficiently large and can be effectively used in various problems. In particular, it allows to define correctly the largest class of Hörmander inner product spaces over a closed compact manifold in such a way that they would not depend on a particular choice of local charts but would possess an interpolation property. Such function spaces are useful in the theory of differential equations, the spectral theory of pseudodifferential operators, the approximation theory, and other areas of modern analysis where Sobolev spaces have already been used.

2. THE MAIN RESULT

Let $1 \leq p \leq \infty$, and let μ be a weight function, that is, there exist numbers $c \geq 1$ and $l > 0$ such that

$$(2) \quad \frac{\mu(\xi)}{\mu(\eta)} \leq c(1 + |\xi - \eta|)^l \quad \text{for each } \xi, \eta \in \mathbb{R}^n.$$

By definition, the Hörmander space $B_{p,\mu}(\mathbb{R}^n)$ consists of all L. Schwartz's distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that their Fourier transforms \widehat{u} are locally Lebesgue integrable in \mathbb{R}^n and $\mu \widehat{u} \in L_p(\mathbb{R}^n)$. The norm in the complex linear space $B_{p,\mu}(\mathbb{R}^n)$ is

$$\|u\|_{B_{p,\mu}(\mathbb{R}^n)} := \|\mu \widehat{u}\|_{L_p(\mathbb{R}^n)}.$$

It is a Hilbert space norm if $p = 2$.

The space $B_{p,\mu}(\mathbb{R}^n)$ is complete with respect to this norm and is embedded continuously in $\mathcal{S}'(\mathbb{R}^n)$. If $1 \leq p < \infty$, then this space is separable and $C_0^\infty(\mathbb{R}^n)$ is dense in it.

Among the Hörmander spaces $B_{2,\mu}(\mathbb{R}^n)$ we only need isotropic spaces. They are corresponded to the radial weight functions $\mu(\xi) = \varphi(\langle \xi \rangle)$, a class of function parameters being defined as follows.

Let *RO* be the class of all Borel measurable functions $\varphi : [1, \infty) \rightarrow (0, \infty)$ for which there exist numbers $a > 1$ and $c \geq 1$ such that

$$(3) \quad c^{-1} \leq \frac{\varphi(\lambda t)}{\varphi(t)} \leq c \quad \text{for each } t \geq 1, \lambda \in [1, a],$$

a and c depending on φ . Such functions are said to be *RO*-varying (or *O*-regularly varying) at $+\infty$. This function class was introduced by V. G. Avakumović in 1936 and has been sufficiently investigated [11, 12]. We recall some its known properties.

Proposition 1. (i) *If $\varphi \in RO$, then both the functions φ and $1/\varphi$ are bounded on every compact interval $[1, b]$ with $1 < b < \infty$.*

(ii) *The following description of the class RO holds:*

$$\varphi \in RO \Leftrightarrow \varphi(t) = \exp\left(\beta(t) + \int_1^t \frac{\varepsilon(\tau)}{\tau} d\tau\right), \quad t \geq 1,$$

where the real-valued functions β and ε are Borel measurable and bounded on $[1, \infty)$.

(iii) *For an arbitrary function $\varphi : [1, \infty) \rightarrow (0, \infty)$ the condition (3) is equivalent to the following: there exist numbers $s_0, s_1 \in \mathbb{R}$, $s_0 \leq s_1$, and $c \geq 1$ such that*

$$(4) \quad t^{-s_0}\varphi(t) \leq c\tau^{-s_0}\varphi(\tau), \quad \tau^{-s_1}\varphi(\tau) \leq ct^{-s_1}\varphi(t) \quad \text{for each } t \geq 1, \tau \geq t.$$

The condition (4) means that the function $t^{-s_0}\varphi(t)$ is equivalent to an increasing function, whereas the function $t^{-s_1}\varphi(t)$ is equivalent to a decreasing one on $[1, \infty)$. Here and below we say that positive functions ψ_1 and ψ_2 are equivalent on a given set if both ψ_1/ψ_2 and ψ_2/ψ_1 are bounded on it; this property will be denoted by $\psi_1 \asymp \psi_2$.

Setting $\lambda := \tau/t$ in the condition (4) we rewrite it in the equivalent form

$$(5) \quad c^{-1}\lambda^{s_0} \leq \frac{\varphi(\lambda t)}{\varphi(t)} \leq c\lambda^{s_1} \quad \text{for each } t \geq 1, \lambda \geq 1.$$

We make the following notations for $\varphi \in RO$:

$$(6) \quad \sigma_0(\varphi) := \sup \{s_0 \in \mathbb{R} : \text{the left-hand side inequality in (5) holds}\},$$

$$(7) \quad \sigma_1(\varphi) := \inf \{s_1 \in \mathbb{R} : \text{the right-hand side inequality in (5) holds}\}.$$

Evidently, $-\infty < \sigma_0(\varphi) \leq \sigma_1(\varphi) < \infty$. The numbers $\sigma_0(\varphi)$ and $\sigma_1(\varphi)$ equal the lower and the upper Matuszewska indices of φ , respectively [12, Sec. 2.2] (Theorem 2.2.2).

Let $\varphi \in RO$. By definition, $H^\varphi(\mathbb{R}^n)$ is the Hilbert space $B_{2,\mu}(\mathbb{R}^n)$ with $\mu(\xi) := \varphi(\langle \xi \rangle)$ for all $\xi \in \mathbb{R}^n$. The inner product in $H^\varphi(\mathbb{R}^n)$ is

$$(u_1, u_2)_{H^\varphi(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \varphi^2(\langle \xi \rangle) \widehat{u}_1(\xi) \overline{\widehat{u}_2(\xi)} d\xi.$$

It induces the norm introduced above.

Note that the space $H^\varphi(\mathbb{R}^n)$ is well defined, since the function $\mu(\xi) = \varphi(\langle \xi \rangle)$ of $\xi \in \mathbb{R}^n$ satisfies (2), i.e., it is a weight function. This will be demonstrated in Section 3.

We also introduce necessary function spaces over Euclidean domains. Let Ω be a domain in \mathbb{R}^n . By definition, the linear space $H^\varphi(\Omega)$ consists of the restrictions $v = u \upharpoonright \Omega$ to Ω of all distributions $u \in H^\varphi(\mathbb{R}^n)$. The norm in $H^\varphi(\Omega)$ is

$$\|v\|_{H^\varphi(\Omega)} := \inf \{ \|u\|_{H^\varphi(\mathbb{R}^n)} : u \in H^\varphi(\mathbb{R}^n), \quad u = v \text{ in } \Omega \}.$$

The space $H^\varphi(\Omega)$ is a separable and Hilbert space with respect to the above norm because it is a quotient space of the separable Hilbert space $H^\varphi(\mathbb{R}^n)$ modulo

$$\{ w \in H^\varphi(\mathbb{R}^n) : \text{supp } w \subseteq \mathbb{R}^n \setminus \Omega \}.$$

If $\varphi(t) = t^s$ with $t \geq 1$ for some $s \in \mathbb{R}$, then $H^\varphi(\Omega)$ coincides with the Sobolev space $H^{(s)}(\Omega)$ of order s .

Let $[X_0, X_1]$ be a couple of complex Hilbert spaces X_0 and X_1 such that $X_1 \subset X_0$ with continuous embedding. A Hilbert space H is called an *interpolation space* with respect to the couple $[X_0, X_1]$ if

- (i) $X_1 \subset H \subset X_0$ with continuous embeddings;
- (ii) an arbitrary linear operator $T : X_0 \rightarrow X_0$, with $T(X_1) \subset X_1$, which is bounded on both spaces X_0 and X_1 is also bounded on H .

Property (ii) implies the following inequality for norms of the operators:

$$\|T\|_{H \rightarrow H} \leq c \max \{ \|T\|_{X_0 \rightarrow X_0}, \|T\|_{X_1 \rightarrow X_1} \},$$

where c is a positive number independent of T [13, Sec. 2.4] (Theorem 2.4.2).

Note that the above properties (i) and (ii) are invariant with respect to a choice of an equivalent norm on H . Therefore we will describe interpolation spaces up to equivalence of norms.

The main result of the paper consists of the following two theorems. Here and below Ω is either the whole space \mathbb{R}^n or a bounded domain in \mathbb{R}^n with an infinitely smooth boundary.

Theorem 1. *Let $-\infty < s_0 < s_1 < \infty$. A Hilbert space H is an interpolation space with respect to the couple of Sobolev spaces $[H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]$ if and only if $H = H^\varphi(\Omega)$ up to norms equivalence for some function parameter $\varphi \in RO$ that satisfies the following conditions:*

- (i) $s_0 \leq \sigma_0(\varphi)$ and, moreover, $s_0 < \sigma_0(\varphi)$ if the supremum in (6) is not attained;
- (ii) $\sigma_1(\varphi) \leq s_1$ and, moreover, $\sigma_1(\varphi) < s_1$ if the infimum in (7) is not attained.

Remark 1. Conditions (i) and (ii) together are equivalent to the bilateral inequality (5), where c is some positive number independent of t and λ .

Let a scale of Hilbert spaces $\{X_s : s \in \mathbb{R}\}$ be such that $X_{s_1} \subset X_{s_0}$ with continuous embedding provided $s_0 < s_1$. A Hilbert space H is called an interpolation space with respect to this scale if H is an interpolation space with respect to a certain couple $[X_{s_0}, X_{s_1}]$ with $s_0 < s_1$.

Theorem 2. *A Hilbert space H is an interpolation space with respect to the Sobolev scale $\{H^{(s)}(\Omega) : s \in \mathbb{R}\}$ if and only if $H = H^\varphi(\Omega)$ up to norms equivalence for some $\varphi \in RO$.*

Remark 2. There are functions $\varphi \in RO$ for which $H^\varphi(\mathbb{R}^n)$ is an intermediate but not an interpolation space with respect to the couple $[H^{(s_0)}(\mathbb{R}^n), H^{(s_1)}(\mathbb{R}^n)]$. See Appendix.

Theorem 2 is a consequence of Theorem 1; both of them will be proved in Section 4.

3. AUXILIARY RESULTS

Here we formulate some necessary results of the Hilbert spaces interpolation theory. It is sufficient to restrict ourselves to separable complex Hilbert spaces.

We say that an ordered couple $[X_0, X_1]$ of Hilbert spaces X_0 and X_1 is *admissible* if these spaces are separable, the continuous and dense embedding $X_1 \hookrightarrow X_0$ holds.

Let us recall the definition of the interpolation of Hilbert spaces with a function parameter. It is a natural generalization of the classical interpolation method of J.-L. Lions and S. G. Krein (see, e.g., [1, Ch. 1, Sec. 2, 5] and [14, Ch. 3, Sec. 10]) to the case where a general enough function is used instead of a number interpolation parameter. The generalization appeared in C. Foiaş and J.-L. Lions' paper [15, Sec. 3.4] and was then studied by several authors.

We denote by \mathcal{B} the set of all Borel measurable functions $\psi : (0, \infty) \rightarrow (0, \infty)$ such that ψ is bounded on each compact interval $[a, b]$ with $0 < a < b < \infty$ and, moreover, $1/\psi$ is bounded on every set $[r, \infty)$ with $r > 0$.

Let a function $\psi \in \mathcal{B}$ and an admissible couple of Hilbert spaces $X = [X_0, X_1]$ be given. For X there exists an isometric isomorphism $J : X_1 \leftrightarrow X_0$ such that J is a self-adjoint positive operator on X_0 with the domain X_1 . The operator J is called a *generating* operator for the couple X . This operator is uniquely determined by X .

An operator $\psi(J)$ is defined in X_0 as the Borel function ψ of J . We denote by $[X_0, X_1]_\psi$ or simply by X_ψ the domain of the operator $\psi(J)$ endowed with the inner product $(u_1, u_2)_{X_\psi} := (\psi(J)u_1, \psi(J)u_2)_{X_0}$ and the corresponding norm $\|u\|_{X_\psi} = \|\psi(J)u\|_{X_0}$. The space X_ψ is Hilbert and separable.

A function $\psi \in \mathcal{B}$ is called an *interpolation parameter* if the following condition is fulfilled for all admissible couples $X = [X_0, X_1]$ and $Y = [Y_0, Y_1]$ of Hilbert spaces and for an arbitrary linear mapping T given on X_0 : if the restriction of T to X_j is a bounded operator $T : X_j \rightarrow Y_j$ for each $j \in \{0, 1\}$, then the restriction of T to X_ψ is also a bounded operator $T : X_\psi \rightarrow Y_\psi$.

If ψ is an interpolation parameter, then we say that the Hilbert space X_ψ is obtained by *interpolation* of the couple X with the function parameter ψ . In this case, we have the dense and continuous embeddings $X_1 \hookrightarrow X_\psi \hookrightarrow X_0$.

The classical result by J.-L. Lions and S. G. Krein consists in that the power function $\psi(t) := t^\theta$ is an interpolation parameter whenever $0 < \theta < 1$, the exponent θ being regarded as a number parameter of the interpolation.

Let us describe the class of all interpolation parameters (in the sense of the above definition).

Let a function $\psi : (0, \infty) \rightarrow (0, \infty)$ and a number $r \geq 0$ be given, then ψ is called *pseudoconcave* on the semiaxis (r, ∞) if there exists a concave function $\psi_1 : (r, \infty) \rightarrow (0, \infty)$ such that $\psi(t) \asymp \psi_1(t)$ for $t > r$. The function ψ is called pseudoconcave on a neighborhood of $+\infty$ if it is pseudoconcave on (r, ∞) , where r is a sufficiently large number.

Proposition 2. *A function $\psi \in \mathcal{B}$ is an interpolation parameter if and only if it is pseudoconcave on a neighborhood of $+\infty$.*

This fact follows from J. Peetre's [16] description of all interpolation functions for the weighted $L_p(\mathbb{R}^n)$ -type spaces (also see [13, Sec. 5.4], Theorem 5.4.4). A proof of Proposition 2 is given in, e.g., [5, Sec. 2.7].

Let $X = [X_0, X_1]$ be an admissible couple of Hilbert spaces. V. I. Ovchinnikov [17, Sec. 11.4] (Theorem 11.4.1) has described (up to equivalence of norms) all the Hilbert spaces that are interpolation spaces with respect to X . In connection with our considerations, his result can be restated as follows.

Proposition 3. *Let H be a Hilbert space. If H is an interpolation space with respect to X , then $H = X_\psi$ up to norms equivalence for some function $\psi \in \mathcal{B}$ being pseudoconcave on a neighborhood of $+\infty$.*

In this connection the following properties of pseudoconcave functions will be of use.

Proposition 4. *Suppose that a function ψ belongs to \mathcal{B} and is pseudoconcave on a neighborhood of $+\infty$. Then there exists a concave function $\psi_0 : (0, \infty) \rightarrow (0, \infty)$ such that $\psi \asymp \psi_0$ on (ε, ∞) for every $\varepsilon > 0$.*

Proof. By the condition, there exists a number $r \gg 1$ and a concave function $\psi_1 : (r, \infty) \rightarrow (0, \infty)$ such that $\psi \asymp \psi_1$ on (r, ∞) . Since the function ψ_1 is concave and positive on (r, ∞) , it increases there. In addition, for every fixed point $t_0 \in (r, \infty)$ the inclination function $(\psi_1(t) - \psi_1(t_0))/(t - t_0)$, $t \in (r, \infty) \setminus \{t_0\}$, decreases. Therefore the function ψ_1 has the right-hand tangent at each point $t_0 > r$, say $t_0 = r + 1$, with the angle between the tangent and the abscissa axis being acute or zero. Let a function $\psi_2(x)$, $x > 0$, be such that its graph coincides with the tangent on $(0, r + 1)$ and with the graph of ψ_1 on $[r + 1, \infty)$. The function ψ_2 increases and is concave on $(0, \infty)$. Set $\psi_0(t) := \psi_2(t) + |\psi_2(0)| + 1$; the function ψ_0 is positive, increases, and is concave on $(0, \infty)$. Chose an arbitrary number $\varepsilon > 0$. Note that $\psi \asymp 1 \asymp \psi_0$ on $[\varepsilon, r + 1 + \varepsilon]$. Since ψ_2 increases and is positive on $[r + 1, \infty)$, we have $|\psi_2(0)| + 1 \leq c \psi_2(t)$ for $t \geq r + 1$, with $c := (|\psi_2(0)| + 1)/\psi_2(r + 1) > 0$. So we arrive at the equivalence $\psi \asymp \psi_1 = \psi_2 \asymp \psi_0$ on $[r + 1, \infty)$. Thus $\psi \asymp \psi_0$ on (ε, ∞) , which is what was to be proved. \square

Proposition 5. *Let a function $\psi \in \mathcal{B}$ and a number $r \geq 0$ be given. The function ψ is pseudoconcave on (r, ∞) if and only if there exists a number $c > 0$ such that*

$$(8) \quad \frac{\psi(t)}{\psi(\tau)} \leq c \max\left\{1, \frac{t}{\tau}\right\} \quad \text{for each } t, \tau > r.$$

Proof. In the $r = 0$ case, this proposition was proved by J. Peetre [16] (also see [13, Sec. 5.4], Theorem 5.4.4), the condition $\psi \in \mathcal{B}$ being superfluous. In the $r > 0$ case, the sufficiency is proved analogously. The necessity is reduced to the $r = 0$ case with the help of Proposition 4.

Indeed, suppose ψ is pseudoconcave on (r, ∞) . Then setting $\varepsilon := r$ in Proposition 4, we have a concave function $\psi_0 : (0, \infty) \rightarrow (0, \infty)$ such that

$$\frac{\psi(t)}{\psi(s)} \asymp \frac{\psi_0(t)}{\psi_0(s)} \leq c_0 \max\left\{1, \frac{t}{s}\right\} \quad \text{for each } t, s > r.$$

(In fact, $c_0 = 1$ for a concave function ψ_0 [16].) □

We also need a reiteration theorem for the interpolation with a function parameter [5] (Theorems 2.1 and 2.3).

Proposition 6. *Suppose that $f, g, \psi \in \mathcal{B}$, and f/g is bounded on a neighborhood of $+\infty$. Let X be an admissible couple of Hilbert spaces. Then the couple $[X_f, X_g]$ is admissible, and $[X_f, X_g]_\psi = X_\omega$ with equality of norms. Here the function $\omega \in \mathcal{B}$ is given by the formula $\omega(t) := f(t) \psi(g(t)/f(t))$ with $t > 0$. Moreover, if f, g , and ψ are interpolation parameters, then ω is an interpolation parameter as well.*

At the end of this section we will prove a result, which justifies that $H^\varphi(\mathbb{R}^n)$ is well defined.

Proposition 7. *Let $\varphi \in RO$; then the function $\mu(\xi) := \varphi(\langle \xi \rangle)$, $\xi \in \mathbb{R}^n$, satisfies (2), i.e., μ is a weight function.*

Proof. Let $\xi, \eta \in \mathbb{R}^n$. By taking squares it is easily to check the inequality $|\langle \xi \rangle - \langle \eta \rangle| \leq ||\xi| - |\eta||$. So in the $\langle \xi \rangle \geq \langle \eta \rangle$ case we have

$$\frac{\langle \xi \rangle}{\langle \eta \rangle} = 1 + \frac{\langle \xi \rangle - \langle \eta \rangle}{\langle \eta \rangle} \leq 1 + |\xi| - |\eta| \leq 1 + |\xi - \eta|.$$

Then by Proposition 1 (iii) we can write

$$\frac{\varphi(\langle \xi \rangle)}{\varphi(\langle \eta \rangle)} \leq c \left(\frac{\langle \xi \rangle}{\langle \eta \rangle} \right)^{s_1} \leq c (1 + |\xi - \eta|)^{\max\{0, s_1\}}.$$

Besides, if $\langle \eta \rangle \geq \langle \xi \rangle$, then

$$\frac{\varphi(\langle \xi \rangle)}{\varphi(\langle \eta \rangle)} \leq c \left(\frac{\langle \xi \rangle}{\langle \eta \rangle} \right)^{s_0} = c \left(\frac{\langle \eta \rangle}{\langle \xi \rangle} \right)^{-s_0} \leq c (1 + |\xi - \eta|)^{\max\{0, -s_0\}}.$$

Thus

$$\frac{\varphi(\langle \xi \rangle)}{\varphi(\langle \eta \rangle)} \leq c (1 + |\xi - \eta|)^l \quad \text{for each } \xi, \eta \in \mathbb{R}^n,$$

with $l := \max\{0, -s_0, s_1\}$. This yields (2) for a certain constant $c \geq 1$. □

4. PROOF OF THE MAIN RESULT

First, we prove two lemmas. In the first one we describe a result of the interpolation of a couple of Sobolev spaces with a function parameter.

Lemma 1. *Let numbers $s_0, s_1 \in \mathbb{R}$ be such that $s_0 < s_1$, and a function $\psi \in \mathcal{B}$ be an interpolation parameter. Set*

$$(9) \quad \varphi(t) := t^{s_0} \psi(t^{s_1-s_0}) \quad \text{for } t \geq 1.$$

Then $\varphi \in RO$ and

$$(10) \quad [H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_\psi = H^\varphi(\Omega)$$

up to equivalence of norms. If $\Omega = \mathbb{R}^n$, then (10) holds with equality of norms.

Proof. First we prove that $\varphi \in RO$. By definition the function φ is Borel measurable on $[1, \infty)$. Let us prove that φ satisfies (3). Since the function ψ is an interpolation parameter, it is pseudoconcave on $(1/2, \infty)$ by Propositions 2 and 4. So, according to Proposition 5 we can write

$$(11) \quad \frac{\varphi(\lambda t)}{\varphi(t)} = \lambda^{s_0} \frac{\psi((\lambda t)^{s_1-s_0})}{\psi(t^{s_1-s_0})} \leq \lambda^{s_0} c \max\{1, \lambda^{s_1-s_0}\} = c \lambda^{s_1},$$

$$(12) \quad \frac{\varphi(t)}{\varphi(\lambda t)} = \lambda^{-s_0} \frac{\psi(t^{s_1-s_0})}{\psi((\lambda t)^{s_1-s_0})} \leq \lambda^{-s_0} c \max\{1, \lambda^{s_0-s_1}\} = c \lambda^{-s_0}$$

for arbitrary $t \geq 1$, $\lambda \geq 1$, and a certain number $c > 0$ that is independent of t and λ . Therefore φ satisfies (3) with $a = 2$, hence belongs to RO .

Let us prove that

$$(13) \quad [H^{(s_0)}(\mathbb{R}^n), H^{(s_1)}(\mathbb{R}^n)]_\psi = H^\varphi(\mathbb{R}^n)$$

with equality of norms. The couple of Sobolev spaces $[H^{(s_0)}(\mathbb{R}^n), H^{(s_1)}(\mathbb{R}^n)]$ is admissible. Let J denote the pseudodifferential operator whose symbol is $\langle \xi \rangle^{s_1-s_0}$, with $\xi \in \mathbb{R}^n$. Then J is a generating operator for the couple. Using the Fourier transform $\mathcal{F} : H^{(s_0)}(\mathbb{R}^n) \leftrightarrow L_2(\mathbb{R}^n, \langle \xi \rangle^{2s_0} d\xi)$ we reduce J to an operator of multiplication by the function $\langle \xi \rangle^{s_1-s_0}$. Hence $\psi(J)$ is reduced to an operator of multiplication by the function $\psi(\langle \xi \rangle^{s_1-s_0}) = \langle \xi \rangle^{-s_0} \varphi(\langle \xi \rangle)$. Therefore we can write the following:

$$\begin{aligned} \|u\|_{[H^{(s_0)}(\mathbb{R}^n), H^{(s_1)}(\mathbb{R}^n)]_\psi}^2 &= \|\psi(J)u\|_{H^{(s_0)}(\mathbb{R}^n)}^2 = \\ &= \int_{\mathbb{R}^n} |(\widehat{\psi(J)u})(\xi)|^2 \langle \xi \rangle^{2s_0} d\xi = \int_{\mathbb{R}^n} |\psi(\langle \xi \rangle^{s_1-s_0}) \widehat{u}(\xi)|^2 \langle \xi \rangle^{2s_0} d\xi = \\ &= \int_{\mathbb{R}^n} \varphi^2(\langle \xi \rangle) |\widehat{u}(\xi)|^2 d\xi = \|u\|_{H^\varphi(\mathbb{R}^n)}^2 \quad \text{for each } u \in C_0^\infty(\mathbb{R}^n). \end{aligned}$$

This implies the equality of spaces (13) as $C_0^\infty(\mathbb{R}^n)$ is dense in both of them. (Note that $C_0^\infty(\mathbb{R}^n)$ is dense in the interpolation space $[H^{(s_0)}(\mathbb{R}^n), H^{(s_1)}(\mathbb{R}^n)]_\psi$ because $C_0^\infty(\mathbb{R}^n)$ is dense in the Sobolev space $H^{(s_1)}(\mathbb{R}^n)$, which is embedded continuously and densely in the interpolation space.)

Now formula (10) for the domain Ω will be deduced from (13). Note that the couple of Sobolev spaces in (10) is admissible. Let R_Ω stand for the operator that restricts distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ to Ω . We have the following operators are bounded and surjective:

$$(14) \quad R_\Omega : H^{(s)}(\mathbb{R}^n) \rightarrow H^{(s)}(\Omega), \quad s \in \mathbb{R},$$

$$(15) \quad R_\Omega : H^\varphi(\mathbb{R}^n) \rightarrow H^\varphi(\Omega).$$

Applying the interpolation with the parameter ψ we infer by (13) that the boundedness of the operators (14), with $s \in \{s_0, s_1\}$, implies boundedness of the operator

$$R_\Omega : H^\varphi(\mathbb{R}^n) = [H^{(s_0)}(\mathbb{R}^n), H^{(s_1)}(\mathbb{R}^n)]_\psi \rightarrow [H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_\psi.$$

Whence since the operator (15) is surjective, we have the inclusion

$$(16) \quad H^\varphi(\Omega) \subset [H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_\psi.$$

Let us prove the inverse inclusion and its continuity. For every integer $k \geq 1$, we need a linear mapping, say T_k , that extends the distribution $u \in H^{(-k)}(\Omega)$ over \mathbb{R}^n and sets the bounded operators

$$(17) \quad T_k : H^{(s)}(\Omega) \rightarrow H^{(s)}(\mathbb{R}^n), \quad \text{with } s \in [-k, k].$$

This operator is given in [2] (Theorem 4.2.2). Chose $k \in \mathbb{N}$ for which $|s_0| < k$, $|s_1| < k$ and consider the operators (17) with $s = s_0$ and $s = s_1$. Since ψ is an interpolation parameter, their boundedness and formula (13) entail boundedness of the operator

$$(18) \quad T_k : [H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_\psi \rightarrow [H^{(s_0)}(\mathbb{R}^n), H^{(s_1)}(\mathbb{R}^n)]_\psi = H^\varphi(\mathbb{R}^n).$$

The product of the bounded operators (15) and (18) gives us the bounded identity operator

$$I = R_\Omega T_k : [H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_\psi \rightarrow H^\varphi(\Omega).$$

So, together with the inclusion (16), we have its continuous inverse. Thus the spaces in (10) coincide; their norms are equivalent due to the Banach theorem on inverse operator. \square

Lemma 2. *Let $s_0, s_1 \in \mathbb{R}$, with $s_0 < s_1$, and $\psi \in \mathcal{B}$. Suppose that φ is defined by (9). Then ψ is an interpolation parameter if and only if φ satisfies (5) with some number $c \geq 1$ which is independent of t and λ .*

Proof. If ψ is an interpolation parameter, then, as we have proved above, the function φ satisfies (11) and (12) for each $t \geq 1$ and $\lambda \geq 1$, i.e., (5) is fulfilled.

Conversely, suppose that φ satisfies (5). Let us prove the inequality (8) for ψ . Considering any $t \geq \tau \geq 1$ and applying the right-hand side of (5) we have

$$\frac{\psi(t)}{\psi(\tau)} = \frac{t^{-s_0/(s_1-s_0)} \varphi(t^{1/(s_1-s_0)})}{\tau^{-s_0/(s_1-s_0)} \varphi(\tau^{1/(s_1-s_0)})} \leq \lambda^{-s_0/(s_1-s_0)} c \lambda^{s_1/(s_1-s_0)} = c \lambda = c \max\left\{1, \frac{t}{\tau}\right\};$$

here $\lambda := t/\tau \geq 1$, whereas the number $c > 0$ does not depend on t and τ . Analogously, considering any $\tau \geq t \geq 1$ and applying the left-hand side of (5) we can write

$$\frac{\psi(t)}{\psi(\tau)} \leq \lambda^{s_0/(s_1-s_0)} c \lambda^{-s_0/(s_1-s_0)} = c = c \max\left\{1, \frac{t}{\tau}\right\},$$

with $\lambda := \tau/t \geq 1$. Thus the inequality (8) holds for $r = 1$. So we conclude by Propositions 2 and 5 that ψ is an interpolation parameter. \square

Proof of Theorem 1. Necessity. Let a Hilbert space H be an interpolation space with respect to the couple of Sobolev spaces $[H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]$. Then, by Propositions 3, 2, and Lemma 1, we have

$$H = [H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_\psi = H^\varphi(\Omega)$$

up to equivalence of norms. Here $\psi \in \mathcal{B}$ is a certain interpolation function parameter, whereas φ is defined by (9). The function φ satisfies (5) in view of Lemma 2. This shows that φ belongs to RO and satisfies both conditions (i) and (ii) in Theorem 1. The necessity is proved.

Sufficiency. Let a function parameter $\varphi \in RO$ satisfy both conditions (i) and (ii) in Theorem 1. Suppose that a Hilbert space H coincides with $H^\varphi(\Omega)$ up to equivalence of norms. Starting with φ let us construct a Borel measurable function ψ such that (9) holds. Namely, we set

$$(19) \quad \psi(\tau) := \begin{cases} \tau^{-s_0/(s_1-s_0)} \varphi(\tau^{1/(s_1-s_0)}) & \text{for } \tau \geq 1, \\ \varphi(1) & \text{for } 0 < \tau < 1. \end{cases}$$

Note that $\psi \in \mathcal{B}$ in view of Proposition 1 (i). The mentioned conditions (i) and (ii) mean that φ satisfies (5). So ψ is an interpolation parameter by Lemma 2. Therefore applying Lemma 1 we have

$$(20) \quad H = H^\varphi(\Omega) = [H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_\psi$$

up to equivalence of norms. Hence H is an interpolation space with respect to the couple $[H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]$. The sufficiency is proved. \square

Proof of Theorem 2. Necessity. Let a Hilbert space H be an interpolation space with respect to the Sobolev scale $\{H^{(s)}(\Omega) : s \in \mathbb{R}\}$. Then H is an interpolation space with respect to a certain couple $[H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]$, with $-\infty < s_0 < s_1 < \infty$. So by Theorem 1 we conclude that $H = H^\varphi(\Omega)$ up to equivalence of norms for some $\varphi \in RO$. The necessity is proved.

Sufficiency. Let $\varphi \in RO$ and a Hilbert space H coincide with $H^\varphi(\Omega)$ up to equivalence of norms. Chose numbers $s_0, s_1 \in \mathbb{R}$ such that $s_0 < \sigma_0(\varphi)$ and $\sigma_1(\varphi) < s_1$. Then conditions (i) and (ii) in Theorem 1 are satisfied. According to this theorem, H is an interpolation space with respect to the couple $[H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]$ and, hence, with respect to the Sobolev scale $\{H^{(s)}(\Omega) : s \in \mathbb{R}\}$. The sufficiency is proved. \square

5. INTERPOLATION PROPERTIES OF HÖRMANDER SPACES

Let us study interpolation properties of the class of Hörmander spaces

$$(21) \quad \{H^\varphi(\Omega) : \varphi \in RO\}.$$

The following theorem shows that every space from the class (21) can be obtained by the interpolation of a couple of Sobolev spaces with an appropriate function parameter.

Theorem 3. *Let $\varphi \in RO$. Chose numbers $s_0, s_1 \in \mathbb{R}$ such that $s_0 < \sigma_0(\varphi) < \sigma_1(\varphi) < s_1$ and define a function ψ by formula (19). Then $\psi \in \mathcal{B}$ is an interpolation parameter, and*

$$(22) \quad [H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_\psi = H^\varphi(\Omega)$$

up to equivalence of norms. If $\Omega = \mathbb{R}^n$, then (22) holds with equality of norms.

Proof. Since φ satisfies (9), this theorem will follow immediately from Lemma 1 if we prove that ψ belongs to \mathcal{B} and is an interpolation parameter. The function ψ is Borel measurable and is bounded on each compact interval $[a, b]$, with $0 < a < b < \infty$. This is true due to Proposition 1. By the choice of s_0 and s_1 the function φ satisfies (5). Specifically, if $t = 1$, then $\varphi(\lambda) \geq c^{-1}\lambda^{s_0}$ for each $\lambda \geq 1$. This yields $\psi(\tau) \geq \min\{\varphi(1), c^{-1}\} > 0$ for $\tau > 0$. Thus $\psi \in \mathcal{B}$. Now it follows from (5) and Lemma 2 that ψ is an interpolation parameter. \square

The following theorem shows that the class of spaces (21) is closed with respect to the interpolation with a function parameter.

Theorem 4. *Let functions $\varphi_0, \varphi_1 \in RO$ and $\psi \in \mathcal{B}$ be given. Suppose that φ_0/φ_1 is bounded on a neighbourhood of $+\infty$ and that ψ is an interpolation parameter. Set*

$$\varphi(t) := \varphi_0(t) \psi\left(\frac{\varphi_1(t)}{\varphi_0(t)}\right) \quad \text{for } t \geq 1.$$

Then $\varphi \in RO$, and

$$(23) \quad [H^{\varphi_0}(\Omega), H^{\varphi_1}(\Omega)]_\psi = H^\varphi(\Omega)$$

up to equivalence of norms. If $\Omega = \mathbb{R}^n$, then (23) holds with equality of norms.

Proof. First we will prove that $\varphi \in RO$. By definition, the function φ is Borel measurable on $[1, \infty)$. Let us check that φ satisfies (3). Since $\varphi_0, \varphi_1 \in RO$, there exist numbers $a, c > 1$ such that

$$(24) \quad c^{-1} \leq \frac{\varphi_j(\lambda t)}{\varphi_j(t)} \leq c \quad \text{for all } t \geq 1, \lambda \in [1, a], j \in \{0, 1\}.$$

It follows from boundedness of the function φ_0/φ_1 on a neighbourhood of ∞ and in view of Proposition 1 (i) that

$$(25) \quad \frac{\varphi_1(t)}{\varphi_0(t)} > \varepsilon \quad \text{for each } t \geq 1,$$

with the number $\varepsilon > 0$ being independent of t . Further, since the function ψ is an interpolation parameter, it is pseudoconcave on (ε, ∞) according to Propositions 2 and 4. This is equivalent, by Proposition 5, to the condition

$$(26) \quad \frac{\psi(\tau)}{\psi(t)} \leq c_0 \max\left\{1, \frac{\tau}{t}\right\} \quad \text{for each } \tau, t > \varepsilon,$$

where the number $c_0 > 1$ does not depend on τ and t . This condition shows that

$$(27) \quad \frac{\psi(t)}{\psi(\tau)} \geq c_0^{-1} \min\left\{1, \frac{t}{\tau}\right\} \quad \text{for each } \tau, t > \varepsilon.$$

Now it follows from (24), (25), and (26) that for every $t \geq 1$ and $\lambda \in [1, a]$ we have

$$\frac{\varphi(\lambda t)}{\varphi(t)} = \frac{\varphi_0(\lambda t)}{\varphi_0(t)} \cdot \frac{\psi(\varphi_1(\lambda t)/\varphi_0(\lambda t))}{\psi(\varphi_1(t)/\varphi_0(t))} \leq c \cdot c_0 \max\left\{1, \frac{\varphi_1(\lambda t)/\varphi_0(\lambda t)}{\varphi_1(t)/\varphi_0(t)}\right\} \leq c^3 c_0.$$

Analogously, it follows from (24), (25), and (27) that

$$\frac{\varphi(\lambda t)}{\varphi(t)} \geq c^{-1} c_0^{-1} \min\left\{1, \frac{\varphi_1(\lambda t)/\varphi_0(\lambda t)}{\varphi_1(t)/\varphi_0(t)}\right\} \geq c^{-3} c_0^{-1}.$$

Thus the Borel measurable function φ satisfies (3), i.e., $\varphi \in RO$.

Now let us deduce (23) from (22) with the help of the reiterated interpolation with a function parameter. Chose numbers $s_0, s_1 \in \mathbb{R}$ such that $s_0 < \sigma_0(\varphi_j)$ and $s_1 > \sigma_1(\varphi_j)$ for each $j \in \{0, 1\}$. By Theorem 3 we have

$$[H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_{\psi_j} = H^{\varphi_j}(\Omega) \quad \text{for each } j \in \{0, 1\}.$$

Here the function $\psi_j \in \mathcal{B}$ is the interpolation parameter defined by formula (19), with φ_j replacing φ . According to Proposition 6 and Lemma 1 we have

$$\begin{aligned} [H^{\varphi_0}(\Omega), H^{\varphi_1}(\Omega)]_{\psi} &= [[H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_{\psi_0}, [H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_{\psi_1}]_{\psi} = \\ &= [H^{(s_0)}(\Omega), H^{(s_1)}(\Omega)]_{\omega} = H^{\varphi}(\Omega). \end{aligned}$$

Here the interpolation parameter $\omega \in \mathcal{B}$ satisfies the equality

$$\omega(\tau) := \psi_0(\tau) \psi\left(\frac{\psi_1(\tau)}{\psi_0(\tau)}\right) = \tau^{-s_0/(s_1-s_0)} \varphi_0(\tau^{1/(s_1-s_0)}) \psi\left(\frac{\varphi_1(\tau^{1/(s_1-s_0)})}{\varphi_0(\tau^{1/(s_1-s_0)})}\right) \quad \text{for } \tau \geq 1;$$

whence

$$\varphi(t) := \varphi_0(t) \psi\left(\frac{\varphi_1(t)}{\varphi_0(t)}\right) = t^{s_0} \omega(t^{s_1-s_0}) \quad \text{for } t \geq 1.$$

The equality of spaces is written up to equivalence of the norms, with the equivalence becoming equality if $\Omega = \mathbb{R}^n$. \square

Thus the class of Hörmander spaces (21) is the maximal extension of the Sobolev scale by an interpolation within the category of Hilbert spaces.

APPENDIX

In connection with Remark 2, we will give an example of a function $\varphi \in RO$ such that

$$(28) \quad H^{(1)}(\mathbb{R}^n) \subset H^\varphi(\mathbb{R}^n) \subset H^{(0)}(\mathbb{R}^n) \quad \text{with continuous embeddings,}$$

$$(29) \quad H^\varphi(\mathbb{R}^n) \text{ is not an interpolation space with respect to } [H^{(0)}(\mathbb{R}^n), H^{(1)}(\mathbb{R}^n)].$$

Set $h(t) := (\log t)^{-1/2} \sin(\log^{1/4} t)$ and define the function

$$\varphi(t) := \begin{cases} t^{h(t)} + \log t & \text{if } t \geq 3, \\ 1 & \text{if } 0 < t < 3. \end{cases}$$

Evidently, $\varphi \in \mathcal{B}$ and $c_0 \leq \varphi(t) \leq c_1 t$ for each $t \geq 1$, where $c_0, c_1 > 0$ do not depend on t . This implies (28) if $\varphi \in RO$. By a simple evaluation we have $t\varphi'(t)/\varphi(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence [11, Sec. 1.2] the function φ is slowly varying at infinity in the sense of J. Karamata, i.e.,

$$(30) \quad \lim_{t \rightarrow \infty} \frac{\varphi(\lambda t)}{\varphi(t)} = 1 \quad \text{for each } \lambda > 0.$$

By Uniform Convergence Theorem [11, Sec. 1.2] (Theorem 1.1) the convergence in (30) is uniform on every compact λ -set in $(0, \infty)$. Therefore $\varphi \in RO$ and, moreover, $\sigma_0(\varphi) = \sigma_1(\varphi) = 0$ [12, Sec. 2.1].

To prove (29) we show that φ is not pseudoconcave on (r, ∞) whenever $r > 0$. Consider the sequences of numbers $t_k := \exp((2\pi k + \pi/2)^4)$ and $s_k := \exp((2\pi k + \pi)^4)$, with $k = 1, 2, 3, \dots$. Calculating we have $h(t_k) = (2\pi k + \pi/2)^{-2}$ and $h(s_k) = 0$, whence

$$\begin{aligned} \log \varphi(t_k) &\geq h(t_k) \log t_k = \left(2\pi k + \frac{\pi}{2}\right)^2, \\ \varphi(s_k) &= 1 + (2\pi k + \pi)^4. \end{aligned}$$

Therefore,

$$\frac{\varphi(t_k)}{\varphi(s_k)} \geq \frac{\exp((2\pi k + \pi/2)^2)}{(1 + (2\pi k + \pi)^4)} \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

But $t_k < s_k$ so, by Proposition 5, the function φ is not pseudoconcave on (r, ∞) whenever $r > 0$.

Now we can prove (29). Suppose the contrary; then by Proposition 3 we have $H^\varphi(\mathbb{R}^n) = [H^{(0)}(\mathbb{R}^n), H^{(1)}(\mathbb{R}^n)]_\psi$ for some function parameter $\psi \in \mathcal{B}$, which is pseudoconcave on a neighbourhood of $+\infty$. Hence $H^\varphi(\mathbb{R}^n) = H^\psi(\mathbb{R}^n)$ by Theorem 4. According to [3, Sec. 2.2] (Theorem 2.2.2) the last equality is equivalent to that $\varphi \asymp \psi$ on $[1, \infty)$. So, φ is pseudoconcave on a neighbourhood of $+\infty$ that leads us to a contradiction.

Thus we have exhibited a function $\varphi \in RO$ satisfying both of (28) and (29). Note that $H^\varphi(\mathbb{R}^n)$ is an interpolation space with respect to the couple $[H^{(-\varepsilon)}(\mathbb{R}^n), H^{(1)}(\mathbb{R}^n)]$ whenever $\varepsilon > 0$. This follows from Theorem 1 because $\sigma_0(\varphi) = \sigma_1(\varphi) = 0$.

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